Incorporating parameter risk into derivatives prices – an approach to bid-ask spreads

June 2013, Warsaw, Poland

Karl F. Bannör
Lehrstuhl für Finanzmathematik
Technische Universität München

This talk is based on joint work with
M. Scherer

Introduction

How to price exotic options?

(a) select a model, e.g.

- Black-Scholes model
- stochastic volatility model
- Lévy driven model
- ...

(b) specify inputs (spot prices, interest rates, vanilla prices)

(c) obtain the model's unobservable parameters

(d) calculate prices of exotic options

- explicit formula
- P(I)DE solving
- Fourier pricing
- Monte Carlo simulation
- ...

Introduction

How to price exotic options?

(a) select a model, e.g.

- Black-Scholes model
- stochastic volatility model
- Lévy driven model
- ...

(b) specify inputs (spot prices, interest rates, vanilla prices)

(c) **obtain the model's unobservable parameters**

(d) calculate prices of exotic options

- explicit formula
- P(I)DE solving
- Fourier pricing
- Monte Carlo simulation
- ...

Introduction

(c) **Obtaining unobservable parameters**

(1) estimation:

$*$ some estimator's value $\hat{\theta}$ is used as "true" parameter θ

$*$ problem: the estimator's volatility and possible bias

(2) calibration to market prices:

$*$ search for parameter that minimizes pricing error, i.e.

$$\theta_0 = \arg\min_{\theta\in\Theta} \eta(\theta), \quad \eta(\theta) = \sum_{\text{vanilla options}} |\text{model price}(\theta) - \text{market price}|$$

$*$ problem: other parameters may fit good as well / local minima

☹ **Problems in (1) and (2)**

$*$ parameter uncertainty

$*$ both procedures disregard information

$*$ not reflected in bid-ask prices

Agenda

Aims

- translate parameter risk into bid-ask spreads
- understand and compare calibration risk of different models / exotic options
- flexible calibration to quoted bid-ask vanilla prices in large class of risk measures

Methodology

- Bayesian approach combined with convex risk measures
- extensive empirical study
- approximation by piecewise linear distortions

Related literature

- Cont (2006): worst-case ansatz for model uncertainty (conservative)
- Lindström (2010): smile modeling by randomizing parameters (non risk-averse)
- Cherny and Madan (2010): parametric calibration ansatz in incomplete markets
- Carr et al. (2001), Branger and Schlag (2004), Xu (2006), Bion-Nadal (2009), ...

Risk and uncertainty

Collection of possible outcomes $(x_\iota)_{\iota \in I}$.

Knight (1921) distinguishes two different situations:

- the probabilities of the outcomes are known, i.e. there is a probability measure on $X := \{x_\iota : \iota \in I\}$

$\Rightarrow$ **risk** according to Knight (1921)

- the probabilities of the outcomes are unknown

$\Rightarrow$ **uncertainty** according to Knight (1921)

Parameter uncertainty...

- $(\Omega, \mathcal{F}, \mathbb{F})$ filtered measurable space
- $(S_t)_{t \geq 0}$ basic instrument
- contingent claims discounted with matching numéraire
- parameterized family of martingale measures $(Q_\theta)_{\theta \in \Theta}$ on $(\Omega, \mathcal{F})$
- parameter $\theta \in \Theta$, risk-neutral value of contingent claim X is

$$\theta \mapsto \mathbb{E}_\theta[X] := \mathbb{E}_{Q_\theta}[X]$$

$\Rightarrow$ parameter uncertainty in the sense of Knight (1921)

...and parameter risk

- $(\Omega, \mathcal{F}, \mathbb{F})$ filtered measurable space
- $(S_t)_{t\geq 0}$ basic instrument
- contingent claims discounted with matching numéraire
- parameterized family of martingale measures $(Q_\theta)_{\theta\in\Theta}$ on $(\Omega, \mathcal{F})$
- parameter $\theta \in \Theta$, risk-neutral value of contingent claim X is

$$\theta \mapsto \mathbb{E}_\theta[X] := \mathbb{E}_{Q_\theta}[X]$$

If additionally:

- probability measure R available on Θ quantifying the likelihood of the parameter $\theta \in \Theta$

$\Rightarrow$ parameter risk in the sense of Knight (1921)

Example: Black-Scholes volatility

Black-Scholes model with risk-free rate $r > 0$ and volatility $\sigma > 0$ follows dynamics

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t$$

How to specify σ? $\Rightarrow$ risk-neutral measures in doubt $(Q_\sigma)_{\sigma\in\mathbb{R}_{>0}}$

- volatility may be estimated from log returns $x_1, \ldots, x_n$ via variance estimator

$$\hat{\sigma}^2{}_N = \frac{1}{\Delta t(N-1)}\sum_{j=1}^{N}(x_j - \bar{x})^2, \quad \bar{x} = \frac{1}{N}\sum_{j=1}^{N} x_j$$

- estimator $\hat{\sigma}^2{}_N$ is χ^2-distributed, i.e. the induced distribution on the parameter space is

$$R(\mathrm{d}x) = \frac{(\Delta t(N-1))^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)(2\sigma_0^2)^{\frac{N-1}{2}}} x^{\frac{N-3}{2}} \exp\left(-\frac{x\Delta t(N-1)}{2\sigma_0^2}\right)\mathbb{1}_{\{x>0\}}\,\mathrm{d}x$$

Risk-capturing functionals

$$\mathcal{D} := \bigcap_{\theta \in \Theta} L^1(Q_\theta) = \text{all admissible derivatives}$$

Economic considerations:

- exotics traders acknowledge parameter uncertainty
- idea: risk $\uparrow$ implies bid-ask spreads $\uparrow$
- Γ risk-capturing functional, X exotic derivative from $\mathcal{D}$
  - ask price: $\Gamma(X)$
  - bid price: $-\Gamma(-X)$
- $\Gamma : \mathcal{D} \to \mathbb{R}$ should fulfill:
  1. order preservation: $X \geq Y \Rightarrow \Gamma(X) \geq \Gamma(Y)$
  2. diversification: $\forall \lambda \in [0,1]$: $\Gamma(\lambda X + (1-\lambda)Y) \leq \lambda\Gamma(X) + (1-\lambda)\Gamma(Y)$
  3. parameter independence consistency:

$$\theta \mapsto \mathbb{E}_\theta[X] \text{ is constant} \Rightarrow \Gamma(X) = \mathbb{E}_\theta[X]$$

Risk-capturing functionals

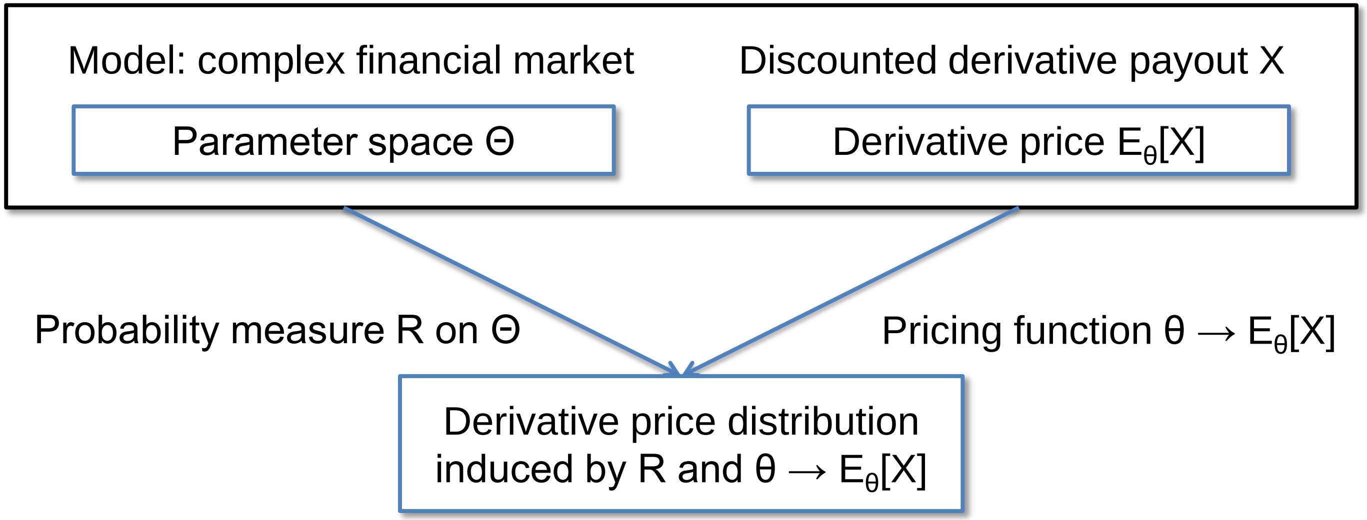
Model: complex financial market

Parameter space Θ

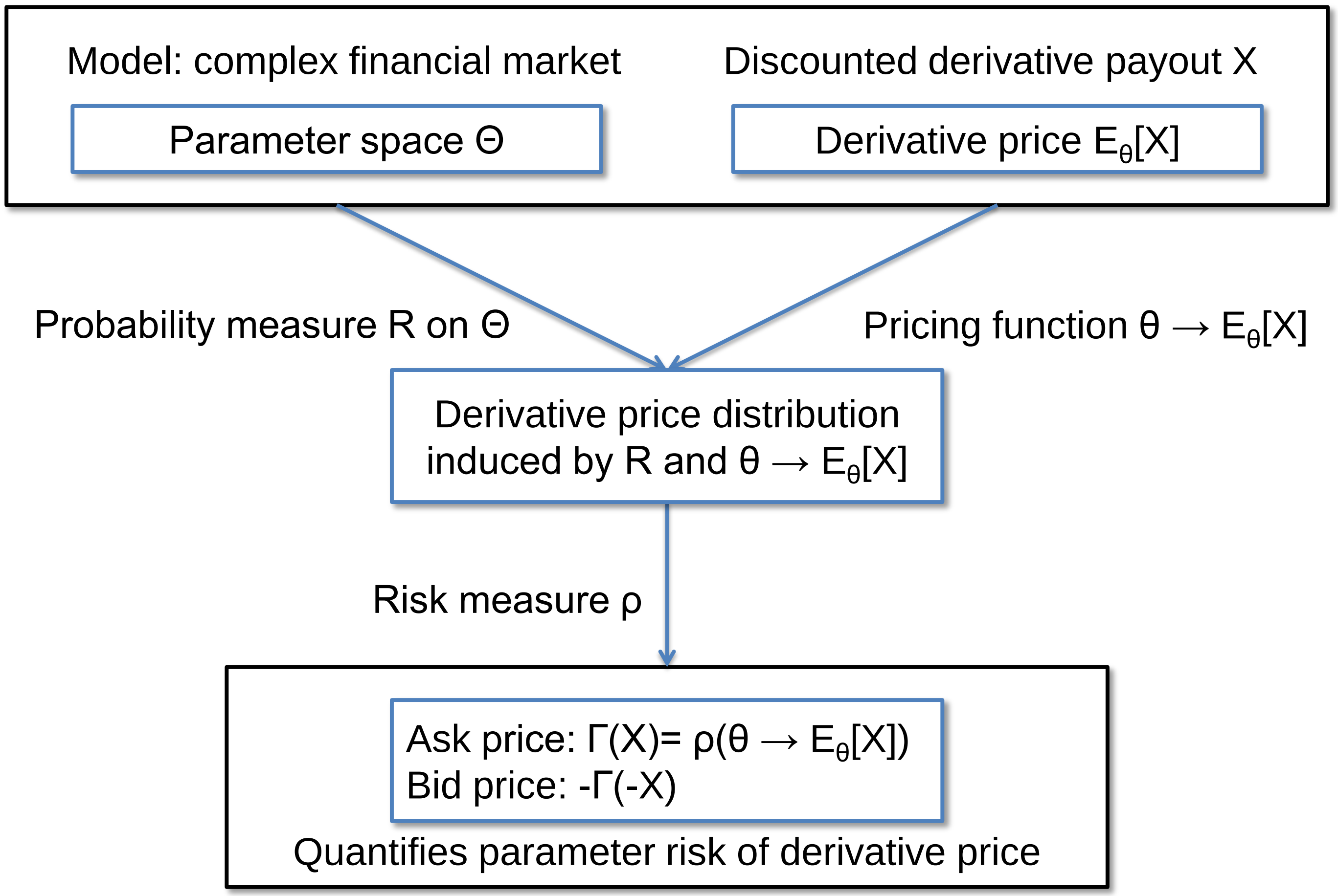
Discounted derivative payout X

Derivative price $E_\theta[X]$

Risk-capturing functionals



Risk-capturing functionals



Risk-capturing functionals

- R probability measure on Θ
- all $\mathcal{A}$-admissible derivatives

$$\mathcal{D}^{\mathcal{A}} := \Big\{ X \in \bigcap_{\theta \in \Theta} L^1(Q_\theta) : \theta \mapsto \mathbb{E}_\theta[X] \in \mathcal{A} \Big\}$$

- ρ convex risk measure (normalized, law-invariant)

Definition 1
The **parameter risk-capturing functional** *w.r.t. ρ is defined by*

$$\Gamma : \mathcal{D}^{\mathcal{A}} \to \mathbb{R}, \quad \Gamma(X) := \rho(\theta \mapsto \mathbb{E}_\theta[X])$$

- $\Gamma(X)$ *is the* **risk-captured ask price** *of* X
- $-\Gamma(-X)$ *is the* **risk-captured bid price** *of* X
- ρ *is the* **generator** *of* Γ

Convergence results

Motivation

- Estimation of parameter $\theta \in \Theta$ by consistent sequence of estimators $(\theta_N)_{N \in \mathbb{N}}$
- Define distributions on Θ by pushforward measures $R_N := P^{N^{\theta_N}}$
- Risk-captured prices converge to price w.r.t. the true parameter, i.e. $\Gamma_{R_N}(X) \to \mathbb{E}_{\theta_0}(X)$, $N \to \infty$?
- If distribution R_N not known or complicated, substitution with asymptotic distribution (e.g. normal) possible?

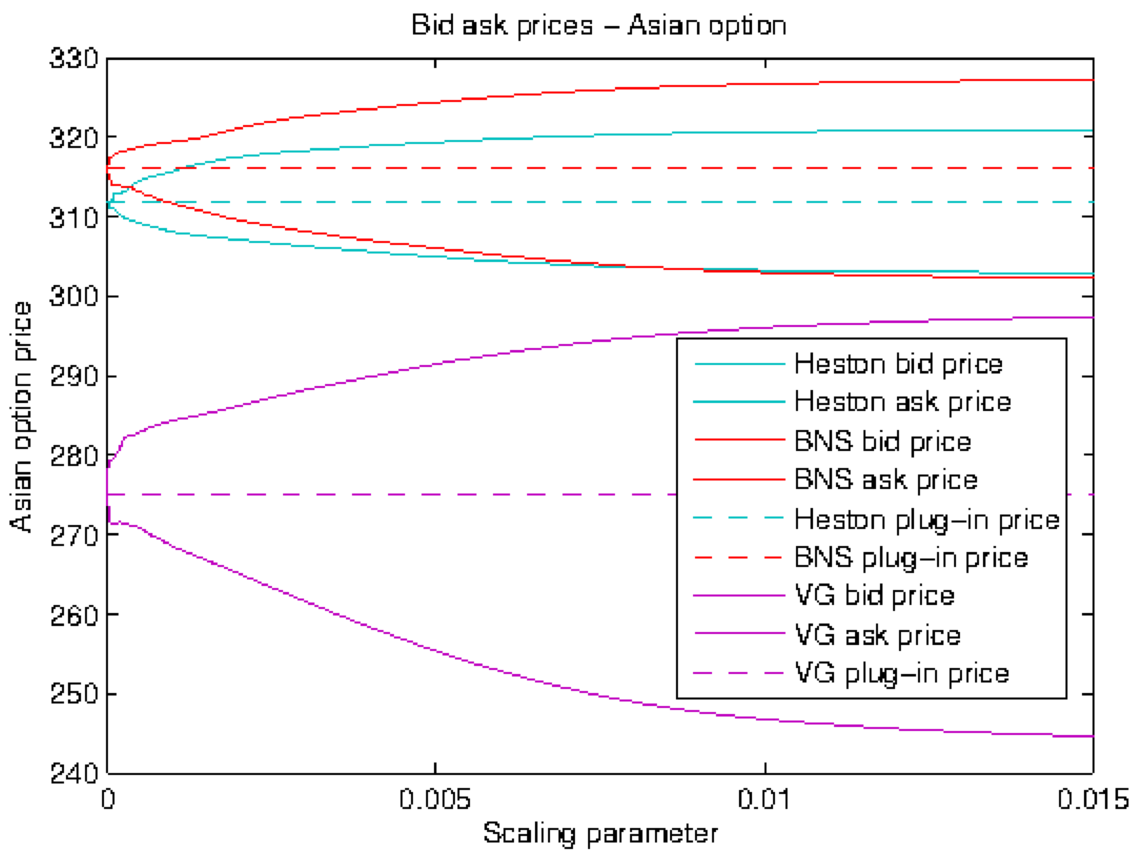
$\Rightarrow$ risk-captured prices should preserve weak convergence of distributions

Convergence results
Spectral risk measures are good-natured

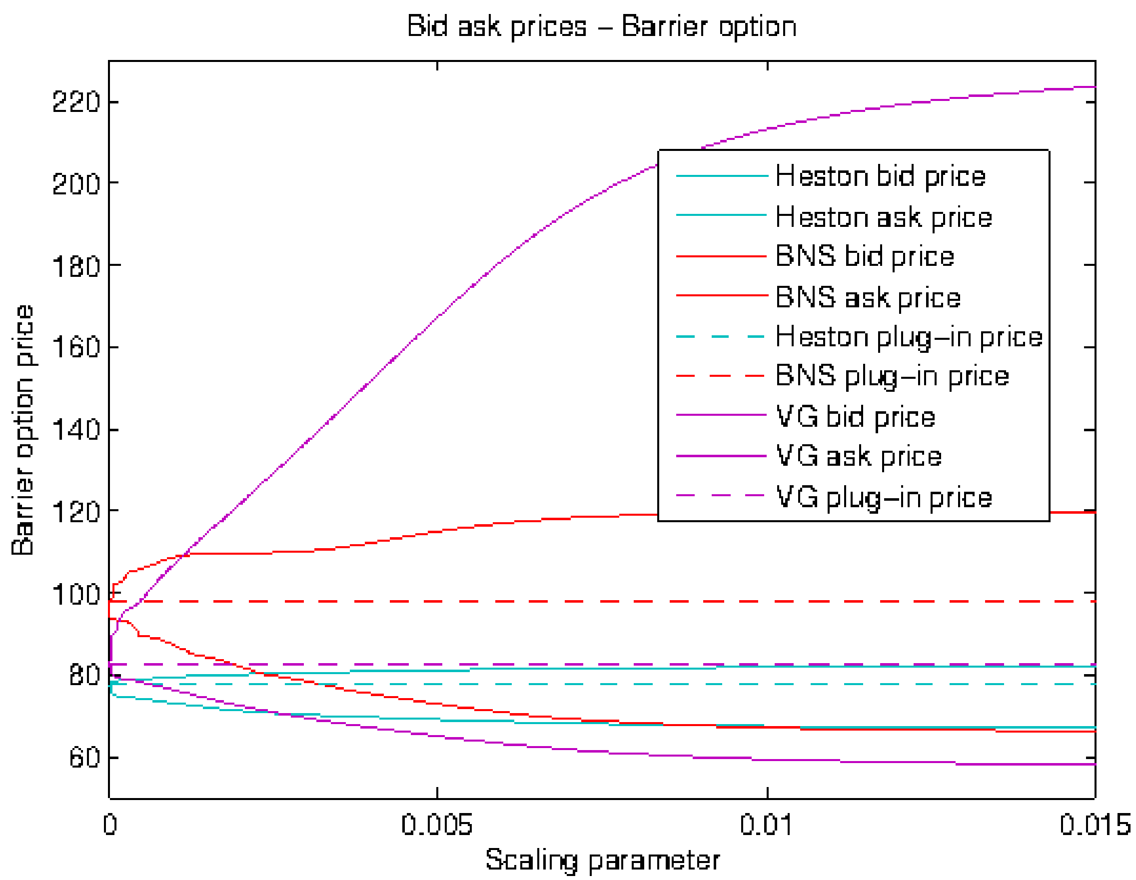
Theorem 2
For spectral risk measures ρ *and* $R_N \to R_0$ *weakly,* $\Gamma^\rho(X; R_N) \to \Gamma^\rho(X; R)$*, if* $\theta \mapsto \mathbb{E}_\theta[X]$ *is continuous and bounded.*

- convergence on $\mathcal{C}^b(\Theta)$ in practice not very restrictive
- risk measure version of Portmanteau theorem as a corollary
- some risk-capturing functionals do not fulfill (CP): essential supremum
- large sample approximations via delta method
- meanwhile, more general results employing stronger topologies are provided by Krätschmer et al. (2012)

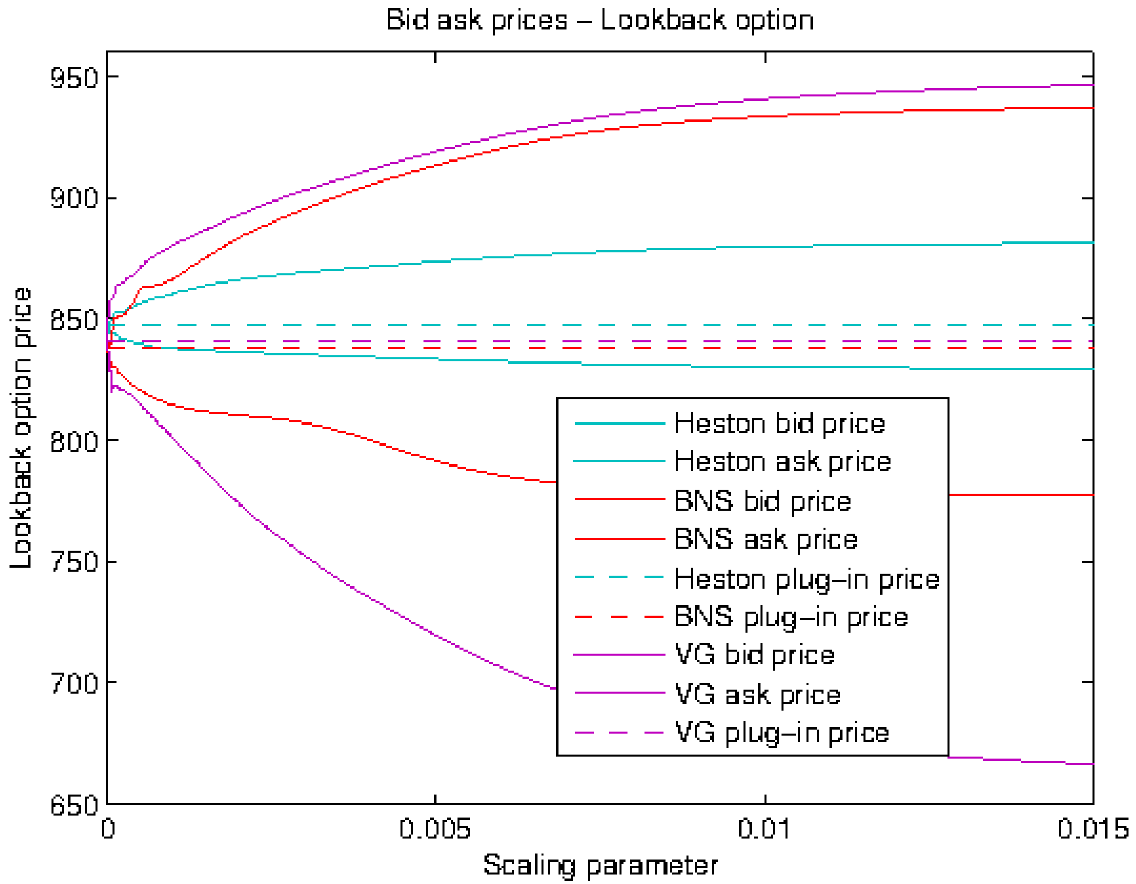
Case study: Asian option bid-ask spreads



Case study: barrier option option bid-ask spreads



Case study: lookback option bid-ask spreads



Choice of risk measure

Which risk-capturing functional Γ, resp. convex risk measure ρ, to choose?

- bid-ask spreads of exotics should match to bid-ask spreads of vanillas
- calibration to vanilla bid-ask prices $\Rightarrow$ market-implied risk measure?

Challenges

- class of all law-invariant convex risk measures may be too big
- description of law-invariant convex risk measures is cumbersome
- flexible and tractable subclass?

Distorted probabilities and the Choquet integral

Definition 3 (Distortion function)
A function $\gamma : [0,1] \to [0,1]$ *is called distortion function, if* γ *is monotone,* $\gamma(0) = 0$*, and* $\gamma(1) = 1$*.*

Natural construction of risk measures employing distortion functions:

Proposition 4 (e.g. Denneberg (1994))
γ *concave distortion function* $\Rightarrow$ *the Choquet integral*

$$\begin{aligned}\Gamma(X) &:= \int X \,\mathrm{d}(\gamma \circ P) \\ &:= \int_{-\infty}^{0} \gamma(P(X > x)) - 1 \,\mathrm{d}x + \int_{0}^{\infty} \gamma(P(X > x)) \,\mathrm{d}x\end{aligned}$$

is a coherent risk measure, the **distortion risk measure with distortion** γ*.*

Why distortion risk measures?

- convenient construction principle for coherent risk measures
- fully characterizable by concave distortion function γ
  $\Rightarrow$ tractable translation of risk measures into functions
- distortion risk measures are the “center of attraction” of coherent risk measures in CLT sense (cf. Belomestny and Krätschmer (2012))
- calculation of Choquet integrals is convenient
- distorted probabilities are widely understood in insurance and finance

Examples for distortion functions

Example 5 (AVaR-family of concave distortions)
$(\mathrm{AVaR}_\alpha)_{\alpha\in(0,1]}$ *is parametric class of distortion risk measures w.r.t. the distortion function*

$$\gamma_\alpha(y) := \begin{cases} \frac{y}{\alpha}, & y \in [0,\alpha] \\ 1, & \textit{otherwise} \end{cases}$$

Example 6 (minmaxvar-family of concave distortions)
$\psi_x(y) : [0,1] \to [0,1]$, $x \in \mathbb{R}_{\geq 0}$ *with*

$$\psi_x(y) := 1 - \left(1 - y^{\frac{1}{x+1}}\right)^{x+1}$$

defines the **minmaxvar***-family of concave distortions (cf. Cherny and Madan (2010))*

Bid-ask calibration problem

- Cherny and Madan (2010): Calibration of bid-ask prices to parametric families of distortion risk measures
- parametric approach feasible?
- little empirical evidence about specific shape of distortion functions $\Rightarrow$ non-parametric approach

Problem 7 (Bid-ask calibration problem)

- $C_1, \ldots, C_M$ *contingent claims with bid-ask quotes* $(\bar{C}_1^{\text{bid}}, \bar{C}_1^{\text{ask}}), \ldots, (\bar{C}_M^{\text{bid}}, \bar{C}_M^{\text{ask}})$
- $\eta : \mathbb{R}_{\geq 0}^{2M} \to \mathbb{R}_{\geq 0}$ *error function*

Convex risk measure $\tilde{\Gamma}$ *solves* **(symmetric) bid-ask calibration problem** *on* $\mathcal{G}$*, if* $\tilde{\Gamma}$ *minimizes the function*

$$\Gamma \mapsto \eta\big(|-\Gamma(-C_1) - \bar{C}_1^{\text{bid}}|, \ldots, |-\Gamma(-C_M) - \bar{C}_M^{\text{bid}}|, \\ |\Gamma(C_1) - \bar{C}_1^{\text{ask}}|, \ldots, |\Gamma(C_M) - \bar{C}_M^{\text{ask}}|\big)$$

over $\mathcal{G}$

Bid-ask calibration problem

Theorem 8 (Existence of a solution to the bid-ask calibration problem)
$K > 0$, η *continuous error function,* $f(X) := \mathbb{E}.[X]$ R*-a.s. bounded and*

$G_K := \{\gamma : [0,1] \to [0,1] : \gamma$ *Lipschitz, concave distortion function with Lipschitz constant* $K\}$.

Then the bid-ask calibration problem has a solution in G_K.

Proof

- *use Arzelà-Ascoli theorem to show* $\|\cdot\|_\infty$*-compactness of* G_K
- *show that the Choquet integral is* $\|\cdot\|_\infty$*-continuous in* γ □

Bid-ask calibration problem
Comments

Class G_K is very wide, since

- every concave distortion function is Lipschitz on $[\varepsilon, 1]$ for every $\varepsilon > 0$
- bid-ask calibration problem can be solved in $\|\cdot\|_\infty$-closed subclasses of distortion functions
- Theorem 10 can be extended to discontinuous distortion functions

Consequences

- bid-ask calibration problem solvable in more general, not-so-easy-to-parameterize classes
- finding a tractable, more flexible class than one-parametric families may yield more flexibility

Non-parametric calibration ansatz

Lemma 9
Every concave distortion function can be $\|\cdot\|_\infty$-approximated by concave piecewise linear distortion functions.

- concave piecewise linear functions are a flexible class
- Choquet integral w.r.t. a concave piecewise linear distorted probability can be easily calculated

$\Rightarrow$
Constrained optimization problem, finding a vector $\Delta\gamma \in \mathbb{R}^N$ minimizing the error function η subject to the constraints

$$\sum_{n=1}^{N} \Delta\gamma_n = 1,\ \Delta\gamma \geq 0,$$

$$\left(\frac{\Delta\gamma_2}{y_2 - y_1} - \frac{\Delta\gamma_1}{y_1 - y_0}, \ldots, \frac{\Delta\gamma_N}{y_N - y_{N-1}} - \frac{\Delta\gamma_{N-1}}{y_{N-1} - y_{N-2}} \right) =: D(\Delta\gamma) \leq 0$$

Non-parametric calibration ansatz

Goal function for optimization:

$$\min_{\Delta\gamma\in\mathbb{R}^N_{\geq 0}} \eta \Bigg(\left|\sum_{n=1}^{N} \Delta\gamma_n \mathbb{E}[f(C_1)|f(C_1)\in[q_{y_{n-1}}, q_{y_n}]] - \bar{C}_1^{\text{bid}}\right|, \dots,$$

$$\left|\sum_{n=1}^{N} \Delta\gamma_n \mathbb{E}[f(C_M)|f(C_M)\in[q_{y_{n-1}}, q_{y_n}]] - \bar{C}_M^{\text{bid}}\right|,$$

$$\left|\sum_{n=1}^{N} \Delta\gamma_n \mathbb{E}[f(C_1)|f(C_1)\in[\text{VaR}_{y_{n-1}}, \text{VaR}_{y_n}]] - \bar{C}_1^{\text{ask}}\right|, \dots,$$

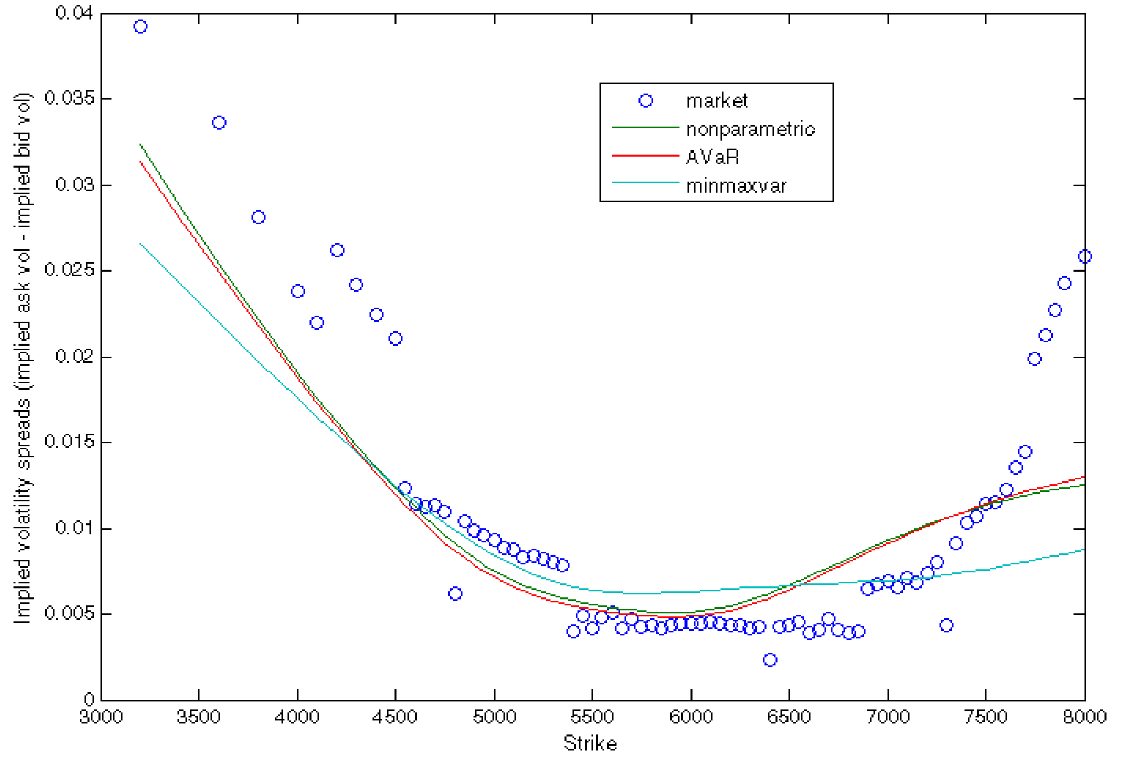
$$\left|\sum_{n=1}^{N} \Delta\gamma_n \mathbb{E}[f(C_M)|f(C_M)\in[\text{VaR}_{y_{n-1}}, \text{VaR}_{y_n}]] - \bar{C}_M^{\text{ask}}\right|\Bigg)$$

Calibration to bid-ask data

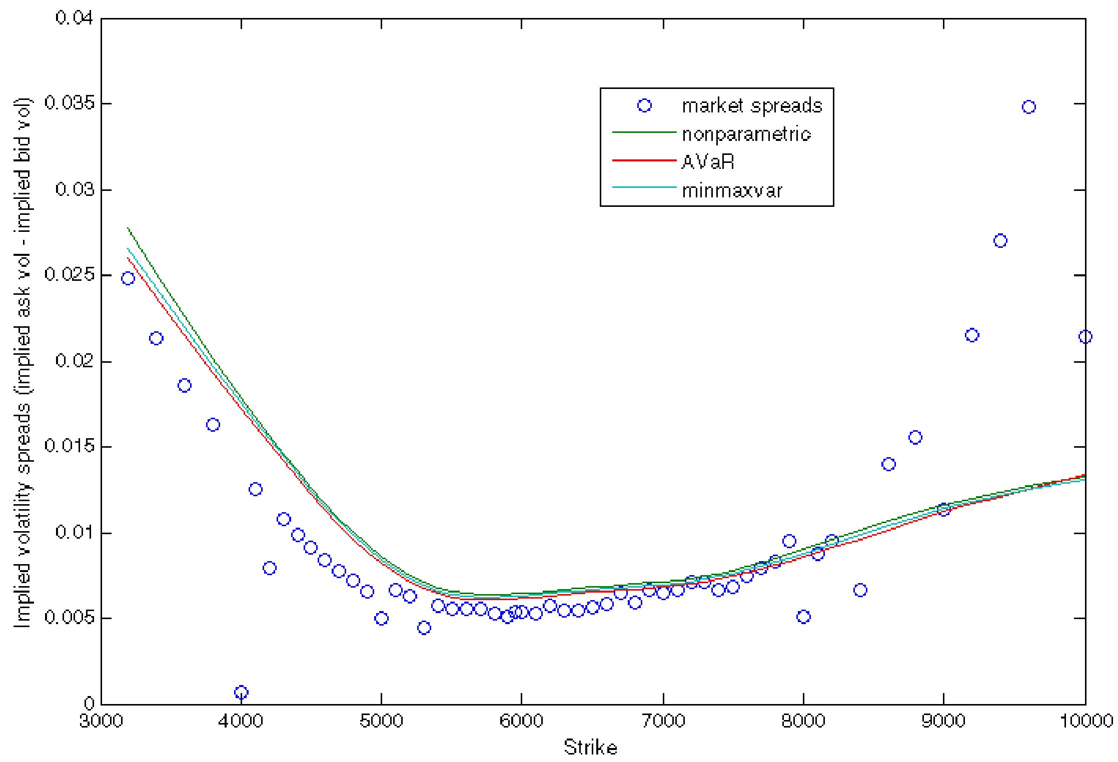
- Data: 501 bid-ask prices of DAX pv calls/puts
- Calibration to mid prices with a BNS model
- Distribution on parameter space as above
- Calibration to bid-ask prices with piecewise linear, AVaR-, **minmaxvar**-distortions

Distortion type	RMSE/mean to bid-ask prices	CPU time
piecewise linear $1\,000$ nodes	1.64%	301.11 sec
piecewise linear 100 nodes	1.64%	4.26 sec
minmaxvar-family	1.65%	3.17 sec
AVaR-family	1.64%	3.73 sec

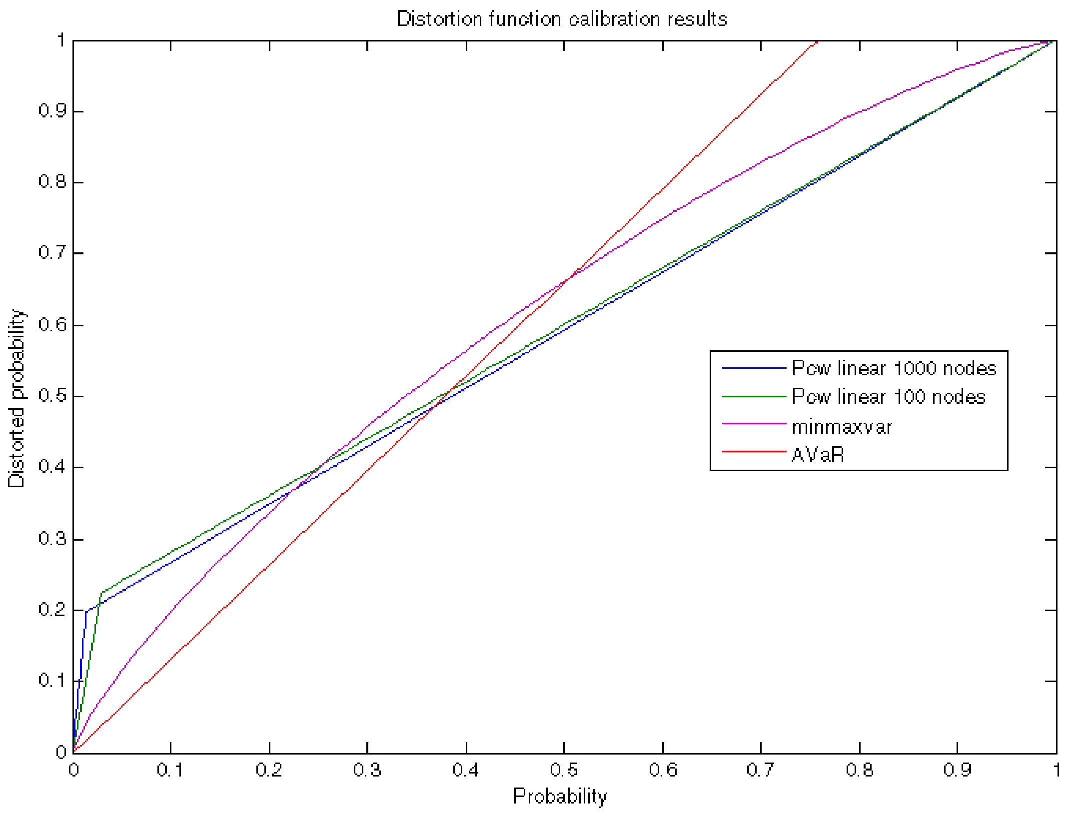
Calibration to bid-ask data



Calibration to bid-ask data



Calibration to bid-ask data



Empirical result

- results are pretty similar in calibration performance
- piecewise linear distortion has characteristic shape
- yields new parametric family allowing for fast and efficient calibration

$$\gamma_\lambda(u) := \begin{cases} 0, & u = 0 \\ \lambda + (1-\lambda)u, & u \in (0,1] \end{cases}$$

is called the $\operatorname{ess\,sup}$-expectation convex combination risk measure with weight $\lambda \in [0,1]$

Distortion framework	RMSE/mean to bid-ask prices	CPU time
piecewise linear 100 nodes	1.64%	4.26 sec
minmaxvar-family	1.65%	3.17 sec
AVaR-family	1.64%	3.73 sec
$\operatorname{ess\,sup}$-exp-family	1.64%	0.21 sec

Conclusion

(1) new methodology for non-linear derivatives pricing

(2) incorporating parameter risk employing convex risk measures

(3) bid-ask spreads of exotic options can be set according to parameter risk

(4) parameter risk of different models can be compared

(5) non-parametric ansatz for the bid-ask calibration problem

(6) empirical results about distortion shape

Literature

(1) K.F. Bannör, M. Scherer, Capturing parameter uncertainty by convex risk measures (2013), *to appear in European Actuarial Journal*

(2) K.F. Bannör, M. Scherer, On the calibration of distortion risk measures to bid-ask prices (2012), *submitted to Quantitative Finance*

(3) J. Bion-Nadal, Bid-ask dynamic pricing in financial markets with transaction costs and liquidity risk, *Journal of Mathematical Economics* **45:11** (2009) pp. 738-750

(4) P. Carr, H. Geman, D. Madan, Pricing and hedging in incomplete markets, *Journal of Financial Economics* **62** (2001), pp. 131–167

(5) A. Cherny, D. Madan, Markets as a counterparty: An introduction to conic finance, *International Journal of Theoretical and Applied Finance* **13:8** (2010) pp. 1149-1177

(6) R. Cont, Model uncertainty and its impact on the pricing of derivative instruments, *Wiley: Mathematical Finance* **16:3** (2006) pp. 519–547

(7) F. Knight, Risk, uncertainty, and profit. *Hart, Schaffner & Marx* (1921).

(8) E. Lindström, Implication of parameter uncertainty on option prices, *Advances in Decision Sciences* **2010** (2010) Article ID 598103

(9) M. Xu, Risk measure pricing and hedging in incomplete markets, *Annals of Finance*, **2** (2006) pp. 51–71

Case study: Heston model (barrier option)